AN EXACT GENERAL INTEGRAL OF THE EQUATIONS OF ONE CLASS OF COMPRESSIBLE FLUIDS

G. I. Nazarov

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 130-131, 1965

Exact general solutions in physical variables that depend on an arbitrary function of two arguments have been found for a certain class of compressible fluids in plane-parallel or axisymmetric flow. These solutions can be useful as a check for numerical methods used in solving boundary-value problems for nonlinear equations of gas dynamics with external conservative forces.

It is known [1] that the motion of a gas in the presence of a magnetic field perpendicular to the plane of flow (or transverse in the case of axisymmetric flow) is described by the system of equations

$$\frac{\partial\varphi}{\partial x}=\frac{1}{\rho y^k}\frac{\partial\psi}{\partial y}, \quad \frac{\partial\varphi}{\partial y}=-\frac{1}{\rho y^k}\frac{\partial\psi}{\partial x}, \quad \frac{v^2}{2}+U+\int\frac{dp}{\rho}+\frac{\mu\rho}{4\pi}=\text{const}, \tag{1}$$

$$p=f(\rho). \tag{2}$$

The notation used here is: φ – velocity potential, ψ – stream function, v – speed, p – pressure, ρ – density, U – potential energy, and μ – a parameter characterizing the magnetic field in the free stream at infinity [2].

When k = 0 Eqs. (1), (2) describe a plane-parallel flow. When k = 1 they represent axisymmetric flow. When $\mu = 0$ the equations reduce to the usual equations of compressible flow. Relation (2) is assumed to be known.

Consider the second problem of dynamics: Given the density (pressure) field $\rho = \rho(x,y)$, find the velocity field and the field of external forces.

The first two equations yield

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{k}{y}\frac{\partial\varphi}{\partial y}+\frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial\rho}{\partial y}\frac{\partial\varphi}{\partial y}\right)=0. \tag{3}$$

Assume

$$\varphi(x,y)=\frac{\Phi(x,y)}{\rho^{1/2}y^{k/2}}. \tag{4}$$

Equation (3) can be reduced to the form

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\left[\frac{k(2-k)}{4y^2}+N(x,y)\right]\Phi=0, \tag{5}$$

where

$$N(x,y)=\frac{1}{2\rho}\left\{\frac{1}{2\rho}\left[\left(\frac{\partial\rho}{\partial x}\right)^2+\left(\frac{\partial\rho}{\partial y}\right)^2\right]-\left(\frac{\partial^2\rho}{\partial x^2}+\frac{\partial^2\rho}{\partial y^2}\right)-\frac{k}{y}\frac{\partial\rho}{\partial y}\right\}.$$

Consider a class of fluids for which $\rho(x,y)$ satisfies the equation

$$\frac{1}{2\rho}\left[\left(\frac{\partial\rho}{\partial x}\right)^2+\left(\frac{\partial\rho}{\partial y}\right)^2\right]-\left(\frac{\partial^2\rho}{\partial x^2}-\frac{\partial^2\rho}{\partial y^2}\right)-\frac{k}{y}\frac{\partial\rho}{dy}=0. \tag{6}$$

For such fluids the function $\Phi(x,y)$ can be found from the equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{k(2-k)}{4y^2}\Phi=0. \tag{7}$$

When k = 0 Eq. (7) reduces to Laplace's equation and, consequently, the function Φ in (4) is determined in the form of a complex potential w(z) (z = x + iy). When k = 1 Eq. (7) is Laplace's equation in cylindrical coordinates for the axisymmetric problem. It is known [3] that in this case the function Φ can also be expressed as a complex potential.

Using (4) and (7), we can determine the speed v. Then, taking into account (2), Bernoulli's equation (1) can be used to determine the potential energy U and hence the field of external conservative forces.

Assuming the solution to (6) to be in the form $\rho = \rho_1(x)\rho_2(y)$, we obtain two nonlinear ordinary equations,

$$2\rho_1\rho_1''-\rho_1'^2\pm\lambda^2\rho_1^2=0, \qquad 2\rho_2\rho_2''-\rho_2'^2+2ky^{-1}\rho_2\rho_2'\mp\lambda^2\rho_2^2=0, \tag{8}$$

in which the terms involving the arbitrary constant λ are to be taken with opposite signs, as indicated.

Assume, initially, $\lambda = 0$ in (8). Integration yields

$$\rho_1 = (a_1 x + b_1)^2, \quad \rho_2 = (a_2 y + b_2)^2 \quad \text{for} \quad k = 0 \quad (a, b = \text{const}), \tag{9}$$

$$\rho_1 = (a_1 x + b_1)^2, \quad \rho_2 = (a_2 \ln y + b_2)^2 \quad \text{for} \quad k = 1. \tag{10}$$

These solutions have real meaning for those boundary-value problems in which the coordinates x, y are bounded.

Relations (9) and (10) can be used to approximate a prescribed density field in given regions of x, y [4]. In that case the constants in (9) and (10) may be regarded as parameters that can be determined at each point of the flow field from the requirement that the function $\rho(x,y)$ and its first derivatives with respect to x and y match the hypothetical expressions for these variables.

In an analogous manner one can obtain the solution for the stream function ψ in the form

$$\psi = \sqrt{y^k \rho}\, \Psi, \qquad \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} - \frac{k(2+k)}{4y^2} \Psi = 0,$$

the density $\rho = \rho_1 \rho_2$ being determined from the same formulas (9), (10).

REFERENCES

1. A. G. Kulikovskii, and G. A. Lyubimov, Magnetohydrodynamics [in Russian], 1962.

2. G. I. Nazarov, "On the exact solutions of some problems of magnetohydrodynamics," Zhurnal prikladnoi mekhaniki i tekhnicheskoi fiziki, no. 2, 1963.

3. G. I. Nazarov, "On an exact general solution of the axisymmetric flow of an incompressible fluid," Trudy Tomskogo universiteta, vol. 163, 1963.

4. G. I. Nazarov, "The method of moving approximations in gas dynamics," Trudy Tomskogo universiteta, vol. 144, 1959.

29 December 1964 Tomsk